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AERODYNAMIC DESIGN AND SIMULATION OF ANNULAR NOZZLE USING METHOD OF CHARACTERISTICS

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ABSTRACT

In this paper we develop a computer code which uses the Method of Characteristics and the Stream Function to define the annular nozzle contour for isentropic, inviscid, irrotational supersonic flows of any working fluid for any user-defined exit Mach number. The contour obtained is compared to theoretical isentropic area ratios for the selected fluid and desired exit Mach number. The accuracy of the nozzle to produce the desired exit Mach number is also checked. The flow field of the nozzles created by the code are independently checked with the commercial Computational Fluid Dynamics (CFD) code ANSYS-FLUENT. ANSYS- FLUENT predictions are used to verify the isentropic flow assumption and that the working fluid reached the user-defined desired exit Mach number. Good agreement in area ratio and exit Mach number is going to be achieved, verifying that the code is accurate.

Key words: Supersonic, Method of Characteristics, Stream Function, Backward Characteristic, Isentropic, Prandtl-Meyer expansion angle.

Nomenclature:

r **-** *Radial coordinate*

ϕ - *angular coordinate*

u - *x direction component velocity*

v - *r direction component velocity*

w - *ϕ direction component velocity*

θ - *Flow Direction*

α - *Mach Angle*

υ - *Prandtl-Meyer Expansion angle*

β - *radius defining the arc of the expansion region*

ψ - *Stream Function*

C_- - *Right running Characteristic*

C_+ -*Left running Characteristic*

INTRODUCTION

Although the burn characteristics of the fuel are an important part of the analysis of a rocket, the rocket's efficiency is primarily dependent upon the nozzle's ability to convert the thermal energy of the fluid to kinetic energy. The main nozzle wall contour plays a critical role in this conversion. It is also important to ensure shocks do not occur within the nozzle. Shocks in the nozzle will disrupt the supersonic flow and will create large losses during the conversion of thermal energy to kinetic energy. The wall contour of the nozzle is the defining factor in whether shocks will or

will not form within the nozzle. The pressure ratio between the chamber and the exit plane of the nozzle dictate the maximum potential Mach number reached by the working fluid.

There are many configurations of supersonic nozzles that will achieve the necessary conversion of thermal energy to kinetic energy to create a rocket's thrust and one among them is the annular nozzle. The Annular nozzle has a curved expansion contour which allows for a higher efficiency for the conversion of thermal energy to kinetic energy. These nozzles tend to be shorter and lighter than the conical nozzle but have the distinct disadvantage of being much more difficult to manufacture and therefore more costly. It is important to note that there are many different variation of the Annular nozzle each having a distinct advantage and disadvantages over another depending on their particular application.

Annular nozzles only have a maximum thrust at a designed altitude which is at some optimized altitude during its flight.

METHOD OF CHARACTERISTICS APPROACH

The method typically used for defining the contour of an annular supersonic nozzle is the Method of Characteristics. In many cases the Method of Characteristics is solved using a Finite-Difference solution. Although there are many variations of this method, the distinct principles of this method remain the same.

The method of characteristics is fully defined by Shapiro and Anderson. Both describe the derivation of the characteristic and compatibility equations as well as explain how to approximate the contour of the nozzle which turns the flow parallel to the nozzle's axisymmetric line with the least amount of losses incurred. They differ by the technique used in defining the contour which turns the flow parallel to the nozzle's axisymmetric line.

Anderson defines the nozzle's contour by calculating line segments between the C_+ Characteristics which are unresolved by a wall point. A C_+ Characteristic is a characteristic that runs to the left as you are looking downstream of the flow. The slope of the line segments defining the wall contour are defined by averaging the flow direction predicted by the two characteristics which the line segment operates between. The intersection of the line segment, i.e. wall contour, and the C_+ characteristic defines the nozzles contour. The line segments always emanate from the end of the previous line segment. For the first line segment, its origination

point is the last point on the known expansion arc. An illustration of this method is given in Figure 2.1 below.

This method becomes more accurate as the number of characteristics used in the calculation increases. The increased number of characteristics also results in a smoother contour. The accuracy of the solution is calculated by comparing the exit area ratio of the calculated nozzle to its ideal isentropic exit area ratio for the desired exit Mach number.
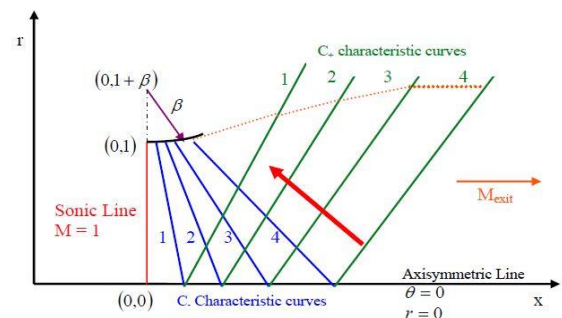

Figure 2.1: Geometry of the Nozzle's Contour Defined by Anderson, 1982

Shapiro takes a slightly different approach in defining the nozzle's contour. He employs a combination of "backward" C_- Characteristics and the Stream Function. A "backward" C_- Characteristic is a C_- Characteristic that is calculated from the last C_+ Characteristic to the first. The direction is indicated by the red arrow in Figure 2.1. The "backward" C_- Characteristics are calculated by assuming that the flow properties along the last C_+ Characteristic remain constant along that characteristic and are the same as the last point on the axisymmetric line. By choosing a point on this C_+ Characteristic as the starting point of the "backward" C_- Characteristic and assuming that a C_+ Characteristic is emanating from the last point on the known expansion arc, a complete calculation of the "backward" C_- Characteristic can be accomplished. Assuming the last point on the known expansion arc is the first point that satisfies the Stream Function of the flow, the calculation along the "backward" C_- Characteristic is continued until the point on the "backward" C_- Characteristic has a radial-component (r-component) greater than the r-component that will satisfy the Stream Function. Once this condition is met, a line segment between the last two points on the "backward" C_- Characteristic can approximate the change in position and flow properties along the "backward" C_- Characteristic. After this line segment is defined, the intersection of the Stream Function emanating from the last calculated streamline point

and the "backward" C_ Characteristic defines the flow properties and position of the next point that satisfies the Stream Function and in turn defines the nozzles contour. The number of "backward" C_Characteristics is increased until the starting point of the "backward" C_ Characteristic on the last C$_+$ Characteristic exhibits an r-component greater than the r-component that would satisfy the Stream Function. In this case, the intersection of the last C$_+$ Characteristic and the Stream Function emanating from the last streamline point defines the position of the last point on the nozzle's contour. Connecting all the points that satisfy the Stream Function with line segments yields the nozzle's contour. Unlike Anderson, Shapiro chose not to use non-dimensional characteristic and compatibility equations. Figure 2.2 illustrates the method Shapiro used to define the nozzle's contour. As with the method outlined by Anderson, Shapiro's method also becomes more accurate by increasing the number of characteristics and "backward" characteristics used. Once again the accuracy is checked by comparing the exit area ratio of the calculated contour with the idealized exit area ratio for an isentropic flow of the desired exit Mach number.

Another example of how the nozzle's contour can be calculated is described in Foelsch, 1949. He describes how the solution to the characteristic equations can be approximated by comparing the conditions of a nozzle to that of a cone. He first deals with equations for a transition curve which converts a conical source flow into a uniform parallel stream of uniform velocity. The equations are obtained by integrating along a Mach line in the region of the conversion of the flow from conical to uniform and parallel. By stepping through Mach numbers by a user-defined increment from 1 to a user-defined exit Mach number, the contour can be established. The last part of the calculation is to resolve the location of the nozzles throat as well as the spherical nature of the sonic line located there. He did this by shifting the location of the conical source flow by a geometric x-component and super imposing a spherical face into a flat face at the throat. The accuracy of the method is again checked by comparing the exit area ratio to the idealized exit area ratio for an isentropic supersonic flow of the desired exit Mach number.
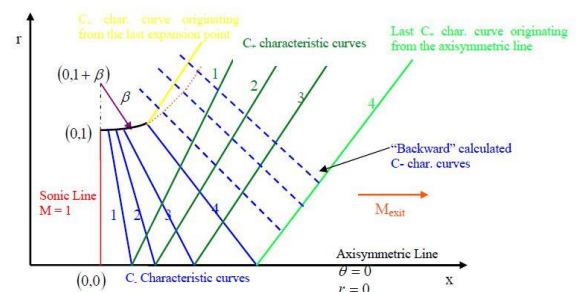

Figure 2.2: Characteristic Geometry Used to Calculate the Wall Contour by Shapiro, 1953-54

DESIGN METHODOLOGY

To implement the characteristic and compatibility equations into a computer code for designing supersonic nozzle contours, the equations for axisymmetric, irrotational, inviscid flow developed must be discretized with boundary conditions defined and applied. The first step in designing a computer code is to discretize the characteristic and compatibility equations. They are rewritten below

$$\left(\frac{dr}{dx}\right)_{char} = tan(\theta \mp \alpha) \qquad ---- 3.1$$

$$d(\theta + \alpha) = \frac{1}{\sqrt{M^2-1}-cot\,\theta}\frac{dr}{r} \qquad\qquad ----3.2(a)$$

$$(along\ c_-characteristic)$$

$$d(\theta - \alpha) = \frac{1}{\sqrt{M^2-1}+cot\,\theta}\frac{dr}{r} \qquad\qquad ----3.2(b)$$

$$(along\ c_+\ characteristic)$$

Equation *3.1* can be split to illustrate the two separate C- and C+ characteristic equations. They are written below

$$\left(\frac{dr}{dx}\right)_{c_-} = tan(\theta - \alpha) \qquad ----3.3\ (a)$$

$$\left(\frac{dr}{dx}\right)_{c_+} = tan(\theta + \alpha) \qquad ----3.3(b)$$

Using the Forward Difference Technique and rearranging equations *3.3a* and *b* yields

$$r_{i+1} - \tan(\theta_i - \alpha_i) . x_{i+1} = r_i - \tan(\theta_i - \alpha_i) . x_i$$

----3.4(a)

(along c_- characteristic)

$$r_{i+1} - \tan(\theta_i + \alpha_i) . x_{i+1} = r_{i+1} - \tan(\theta_i + \alpha_i) . x_{i+1}$$

----3.4(b)

(along c_+ characteristic)

Note that all variables with subscript I are known quantities and variables with subscript $i+1$ are unknown quantities. Equations *3.4a* and *3.4b* are the discretized characteristic equations that will define the location in the x-r space where the C- and C+ characteristics curves intersect. This collection of points is called the Characteristic Net.

Equation *3.2a* and *3.2b*, the compatibility equations, can also be discretized. Using the Forward Difference Technique and rearranging gives

$$(\theta_{i+1} + v_{i+1}) = (\theta_i + v_i) + \frac{1}{\sqrt{M^2 - 1} - \cot\theta} \frac{r_{i+1} - r_i}{r_i}$$

----.3.5(a)

(along c_- characteristic)

$$(\theta_{i+1} - v_{i+1}) = (\theta_i - v_i) - \frac{1}{\sqrt{M^2 - 1} - \cot\theta} \frac{r_{i+1} - r_i}{r_i}$$

----3.5(b)

(along c_+ characteristic)

Note that r_{i+1} is on the right side of the compatibility equations. In the calculation, this is a known quantity from the solution of the Characteristic Net, equations *3.4a* and *b*. Therefore, the compatibility equations can be solved simultaneously to find the direction of the flow and the Prandtl-Meyer Expansion angle at the point where the characteristics intersect. The speed, Mach number, and Mach angle of the flow at that point of intersection can be extrapolated from the Prandtl-Meyer expansion angle given as

$$v = \sqrt{\frac{\gamma-1}{\gamma+1}} \tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}(M_i^2 - 1)}\right) - \tan^{-1}\left(\sqrt{M_i^2 - 1}\right)$$

----3.9

Stream Function Equation and Discretation:
Since in steady, axisymmetric flow there are only two space coordinates, the statement of the continuity equation is the necessary and sufficient condition for the existence of a Stream Function.
The most generic form of the Stream Function is

$$\Psi = f(x, r) \quad ----3.10$$

and ψ is a constant. To satisfy the streamline condition, there is no change in the constant.
Therefore, $d\psi = 0$. Differentiating equation *3.10* and substituting $d\psi = 0$, gives

$$\frac{\partial\psi}{\partial x} dx + \frac{\partial\psi}{\partial r} dr = 0 \quad ----3.11$$

Equation *3.11* is valid for small changes in *dx* and *dr*. To satisfy the continuity equation at a point, ψ is defined by

$$v = \frac{\rho_0}{\rho} \frac{1}{r} \frac{\partial\Psi}{\partial x} \quad ----3.12(a)$$

$$u = \frac{\rho_0}{\rho} \frac{1}{r} \frac{\partial\Psi}{\partial r} \quad ----3.12(b)$$

Solving equations *3.12a* and *3.12b* for $\partial\psi/\partial x$ and $\partial\psi/\partial r$, respectively, and substituting into equation *3.11* yields

$$-rv\frac{\rho}{\rho_0} dx + ru\frac{\rho}{\rho_0} dr = 0 \quad ----3.13$$

Since *u* and *v* are not explicitly known at the points in a supersonic flow field using the non-dimensional Method of Characteristics, they must be converted to their associated Mach number at the point through the following transformation. From geometry,

$u = \vec{v}\cos\theta$ ----3.14(a)

$v = \vec{v}\sin\theta$ ----3.14(b)

The velocity and Mach number at the point is known through the following relation

$\vec{v} = a.M$ ----3.15

Substituting equation *3.15* into equation *3.14a* and *3.14b* gives

$u = a.M\cos\theta$ ----3.16(a)

$v = a.M\sin\theta$ ----3.16(b)

Substituting these results into equation *3.13* yields

$-ra.M\sin\theta \frac{\rho}{\rho_0}dx + ra.M\cos\theta \frac{\rho}{\rho_0}dr = 0$ ----3.17

Since the equation is satisfied at a point, *r, a, M* and ρ/ρ_o can be divided out, simplifying equation *3.17* to

$r_{i+1} - \tan\theta_i.x_{i+1} = r_i - \tan\theta_i.x_i$ ----3.18

Equation *3.18* illustrates that a steady, irrotational, supersonic flow's streamline is dependent only on the direction of the flow and location of its previous point. In conjunction with an approximation by a straight line between the point where the r-coordinate is greater that the r-coordinate of the last known wall contour point and the previous point along the "backward" calculated C- characteristic, equation *3.18* can be employed to find the location of the next point on the streamline by the solution of two intersecting lines.

Calculating the Point Satisfying the Stream Function:

The last step in satisfying the Stream Function equation is to develop straight line approximations for the flow property changes between the last two points along the "backward" C- characteristic. The general equation for a straight line

$r = mx + z$ ----3.19

where *m* is the slope, *z* is the r-intercept, *r* is the r-coordinate and *x* is the x-coordinate of a point on the line. Since the positions, (*x, r*), of two points are known along the approximation line, the slope and r-intercept of the line can be solved using the system of equations.

$r_1 = mx_1 + z$ ----3.20(a)

$r_2 = mx_2 + z$ ----3.20(b)

Since the Stream Function is also dependent on the flow direction at the previous point, the change in θ along the straight line approximation must also be calculated. Modifying equation 3.19 for θ on the θ − r plane yields

$\theta = m_\theta x + z_\theta$ ----3.21

Where m_θ is the slope, z_θ is the θ-intercept, θ is the θ-coordinate and *x* is the x-coordinate of a point on the line.

Since the positions, (*x*, θ), of two points are known along the approximation line, the slope and θ-intercept of the line can be solved using the system of equations

$\theta_1 = m_\theta x_1 + z_\theta$ ----3.22(a)

$\theta_2 = m_\theta x_2 + z_\theta$ ----3.22(b)

Now that the equations for the approximation lines are known, the intersection of the Stream Function equation with the (x, r) space approximation line of the "backward" C- characteristic yields the solution of the position for the next point that satisfies the Stream Function. The x-component of the solution point that satisfies the Stream Function can be used in the (x, θ) space line approximation equations to find the flow direction at the solution streamline point. This is illustrated in equation 3.23.

$\theta_{i+1} = m_\theta x_{i+1} + z_\theta$ ----3.23

This solution wall contour point is used in the calculation of the next streamline point for the next "backward" calculated C- characteristic.

RESULTS AND DISCUSSION

For cost effective design, it is advantageous to validate the nozzles' contour in a Computational Fluid Dynamics (CFD) code such as ANSYS-FLUENT. This step in the design process can save money. It allows the designer to see if the flow reaches the desired exit velocity and if shocks develop in the flow without the need for

materials or time for testing. Below is an outline of the steps taken to validate the nozzle contour developed from the theory above. For simulation convergence, continuity, x-velocity, y-velocity and energy changes were all required to be less than 0.001 which is ANSYS-FLUENT's default parameter.

4.1. Theoretical Accuracy of Computer Code:

The first check of accuracy for the program was comparing the desired exit Mach number with the exit Mach number calculated by the program. Table 4.1.1 below shows the percent difference between the desired and computer calculated exit --------Mach numbers. Table 4.1.1 also shows how the code becomes more accurate as a smaller change in Prandtl-Meyer expansion angle is used during calculations.

Since the equations were based on isentropic flow theory, the accuracy of the code was also checked by calculating the exit to throat area ratio using equation *4.1* substituting in the user- defined ratio of specific heats and computer calculated exit Mach number. This yields the theoretical area ratio for the Mach number actually calculated by the program.

$$\frac{A_{exit}}{A_{throat}} = \sqrt{\frac{1}{Ma^2_{exit}} \cdot \left[\frac{2}{\gamma+1} \cdot \left(1 + \frac{\gamma-1}{2} \cdot Ma^2_{exit}\right)\right]^{\frac{\gamma+1}{\gamma-1}}}$$

---*4.1*

The theoretical and computer calculated isentropic area ratios for the desired exit Mach number were also compared for a user-defined ratio of specific heats in Table 4.1.1. The output of the MATLAB code is shown in the figure below   and   the   contour   has   been   developed   for   various   input   results

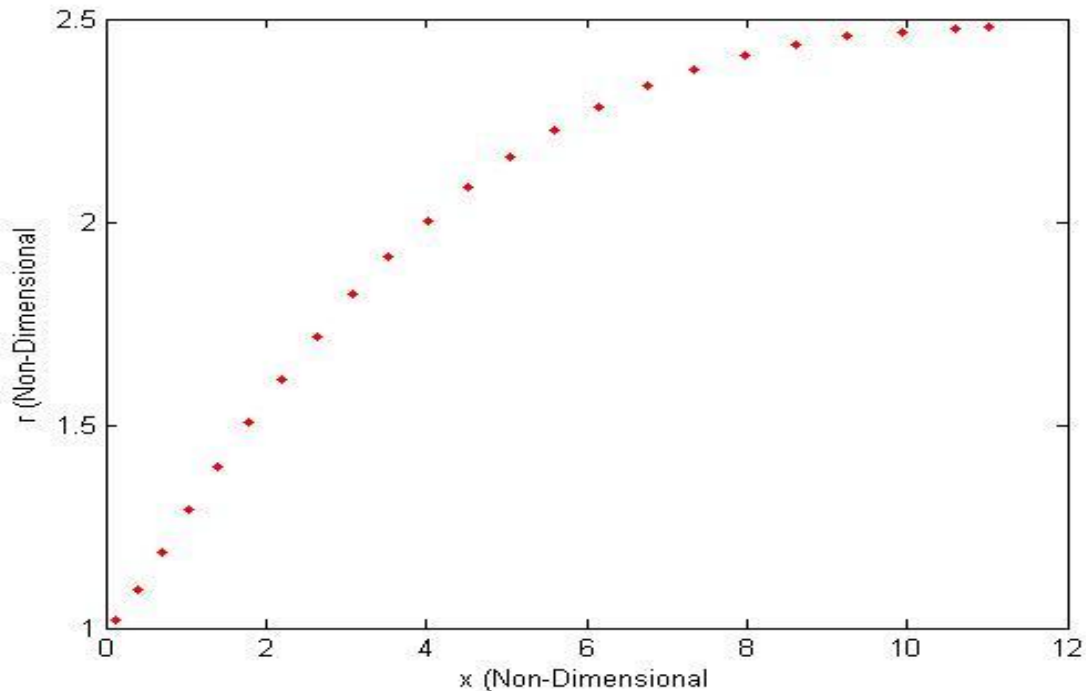| Table 4.1.1	Code Accuracy check for $\gamma=1.4$ $Ma=3.0$ $\beta=1.0, r_{throat}/r_{throat}=1.0$ (Dimensionless)						
	$\left[\frac{A_{exit}}{A_{throat}}\right]_{comp}$	$\left[\frac{A_{exit}}{A_{throat}}\right]_{Theory}$	$\left[\frac{A_{exit}}{A_{throat}}\right]_{\%error}$	Ma_{comp}	$Ma_{\%error}$	$\left[\frac{A_{exit}}{A_{throat}}\right]_{comp,theory}$	$\left[\frac{A_{exit}}{A_{throat}}\right]_{comp,\%error}$
$\Delta\upsilon=0.05$							
Annular	5.6588	4.2346	33.63%	3.2489	8.30%	5.3635	5.51%
$\Delta\upsilon=0.025$							
Annular	5.5827	4.2346	31.84%	3.2455	8.18%	5.3463	4.42%
$\Delta\upsilon=0.01$							
Annular	4.4882	4.2346	5.99%	3.0289	0.96%	4.3527	3.11%
$\Delta\upsilon=0.005$							
Annular	4.4938	4.2346	6.12%	3.0323	1.08%	4.3668	2.91%
$\Delta\upsilon=0.0025$							
Annular	4.3940	4.2346	3.76%	3.0093	0.31%	4.2722	2.85%

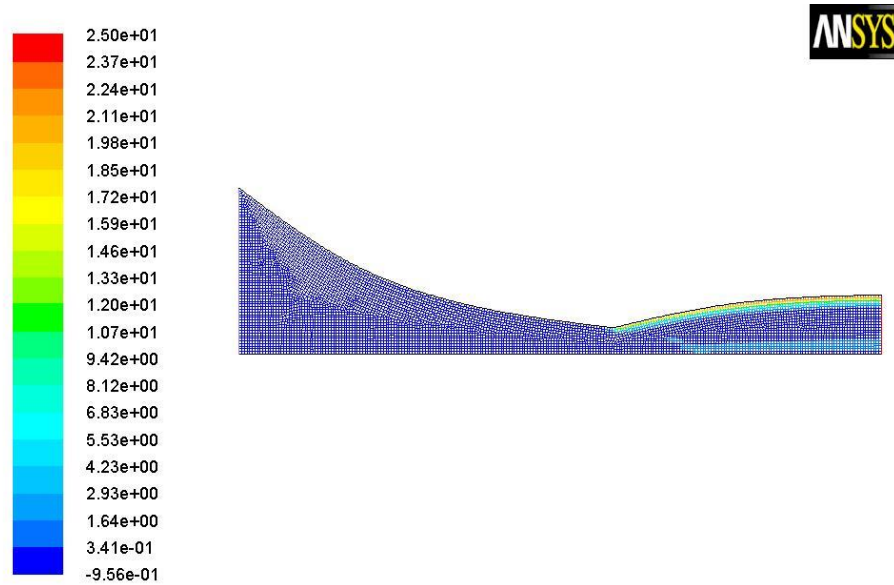*Figure 4.1.1: Plot of the Annular Nozzle Contour*

Ansys – Fluent Simulations:

In order to run a simulation of the flow in supersonic annular nozzles, the nozzle is built virtually so that a mesh can be generated in the fluid region and the program developed produces a set of points to define the annular nozzles contour. Figure 4.2.1 shows a typical dimensional entropy contour for an annular nozzle.

From Figure 4.2.1, it can be seen that the large majority of the fluid domain demonstrates constant entropy signifying that the isentropic flow assumption is valid. The region near the wall contour where the entropy is changing is a result of the discontinuities in the wall contour. Since the wall contour was defined by a set of points that were connected by straight line segments, it is discontinuous at the points that connect them. The change in entropy in the flow field is a propagation of these discontinuities.

The exit Mach number is checked by the Mach contours of the simulation as well as having ANSYS-FLUENT calculate the area-weighted Mach number at the exit plane of the nozzle. The area weighted Mach number calculated by ANSYS-FLUENT is compared to the Mach number calculated by the program and the desired exit Mach number. Figure 4.2.2 shows the typical Mach contours of an annular nozzle designed for a Mach number of 3.0.

From Figure 4.2.2, it is clear that the Mach number at the exit has a maximum of 3.01. Table 4.2.1 shows typical % error in exit Mach number for annular nozzles designed by the program detailed in Section 3.0 when compared to FLUENT results. After comparing the Mach numbers, it is clear that the code developed is valid and accurate.

Figure 4.2.1: Typical Entropy Contour for an Annular Nozzle
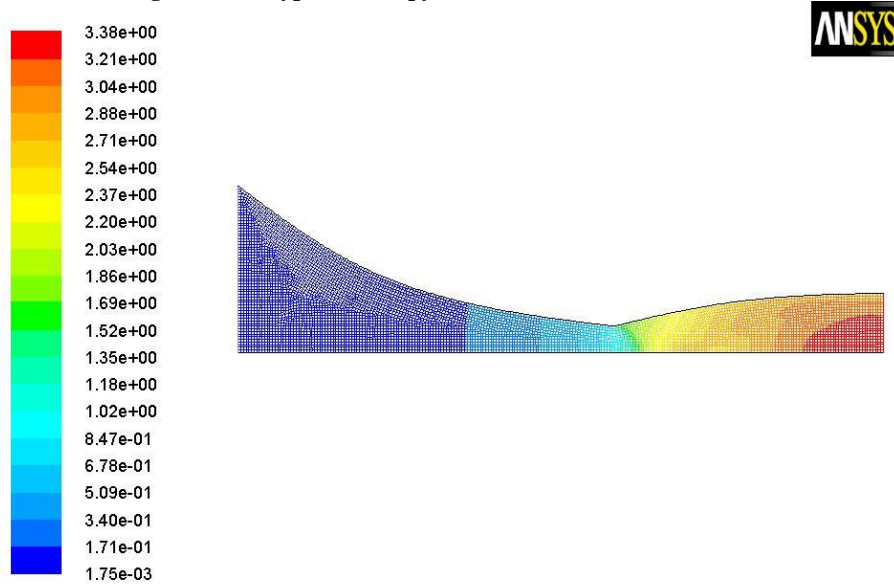
Figure 4.2.2: Typical Mach Contours for an Annular Nozzle

Table 4.2.1		Mach Number Comparisons for Annular Nozzle	
	$Ma_{Desired}$	$Ma_{ComputerCalculated}$	$Ma_{FluentCalculated}$
	3.0	3.0062	2.9528
Percent Error From $Ma_{Desired}$		0.21%	1.57%

CONCLUSION

The code developed proves to be a useful tool in creating annular nozzle contour for isentropic, irrotational, inviscid flow. The program exhibits increasing accuracy in the exit Mach number and exit area ratio as the incremental Prandtl- Meyer expansion angle decreases. This accuracy increase is independent of fluid or desired exit Mach number. The exit Mach number of the nozzles calculated with the program shows good agreement with the ANSYS-FLUENT simulated exit Mach numbers. This independently confirms the accuracy of the program in calculating supersonic nozzle contours for inviscid, isentropic, irrotational supersonic flows.

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